

Periodic Chandrasekhar recursions

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Abstract

This paper extends the Chandrasekhar-type recursions due to Morf, Sidhu, and Kailath "Some new algorithms for recursive estimation in constant, linear, discrete-time systems, *IEEE Trans. Autom. Control* 19 (1974) 315-323" to the case of periodic time-varying state-space models. We show that the S -lagged increments of the one-step prediction error covariance satisfy certain recursions from which we derive some algorithms for linear least squares estimation for periodic state-space models. The proposed recursions may have potential computational advantages over the Kalman Filter and, in particular, the periodic Riccati difference equation.

Keywords: Periodic state-space models, Chandrasekhar-type recursions, Kalman Filter, periodic Riccati difference equation.

INTRODUCTION

Morf et al (1974) proposed recursions that substitute the Kalman Filter for linear least squares estimation of discrete-time time-invariant state space models, with a simpler computational complexity. The new algorithms have been called Chandrasekhar-type recursions because they are analog to certain differential equations encountered in continuous-time problems (Kailath, 1973). Since there, a considerable attention has been paid in the three recent decades to the Chandrasekhar-type recursions (see e.g. Friedlander et al, 1978; Morf and Kailath; 1975; Houacine and Demonent, 1986; Houacine, 1991; Nakamori et al, 2004; Nakamori, 2007). At present, there exist several

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useful applications of the Chandrasekhar filter in improving computational aspects related to the building of linear time-invariant models. We mention non exhaustively the likelihood evaluation (see Pearlman, 1980; Mélard, 1984; Kohn and Ansley, 1985 for autoregressive moving average, *ARMA*, models and Shea, 1989 for vector *ARMA* models), the calculation of the exact Fisher information matrix (see Mélard and Klein (1994) for the *ARMA* case and Klein et al (1998) for general dynamic time-invariant models), and the development of fast variants of the recursive least squares algorithm (Houacine, 1991; Sayed and Kailath, 1994; Nakamori et al 2004). As is well known, the Chandrasekhar equations are restricted to the case of time-invariant state-space models because of their particular time invariance structure and it seems that there is no results tied to the class of all nonstationarity, except in very special cases (Sayed and Kailath, 1994). A particular class of nonstationarity whose importance has no need to be proven is the one of periodic linear models. Important progress has been made recently in the building and analysis of periodic *ARMA* (*PARMA*) and periodic state-space characterizations. The objective was to develop extensions of similar methods for standard time-invariant models to their periodic counterparts, without transforming periodic systems to their corresponding multivariate time-invariant representations in order to simplify the computational burden. Despite the current abundance of computational methods for periodic state-space models (see e.g. Lund and Basawa, 2000; Varga and Van Dooren, 2001; Gautier, 2005; Bentarzi and Aknouche, 2005; Aknouche, 2007; Aknouche and Hamdi, 2007 Aknouche et al, 2007 and the references therein) it seems that there is no results concerning extensions of the Chandrasekhar recursions to the periodic case. This paper proposes some algorithms for linear least squares estimation of periodic state-space models. Our methods extend the Chandrasekhar algorithms proposed by Morf et al (1974) to the periodic time-varying case and retain their desirable features. As a result, the periodic Chandrasekhar recursions are used through the innovation approach to efficiently evaluate the likelihood of periodic *ARMA* models.

The rest of this paper is organized as follows. Section 1 briefly recalls some preliminary defini-

tions and facts about periodic state-space models and their corresponding Kalman Filter. In Section 2 we develop some Chandrasekhar-type algorithms that substitute the Kalman filter for periodic state-space models. The initialization problem will be studied in Section 3.

I. PRELIMINARY DEFINITIONS AND NOTATIONS

Consider the following linear periodic state-space model

$$\begin{cases} \mathbf{x}_{t+1} = F_t \mathbf{x}_t + G_t \epsilon_t \\ \mathbf{y}_t = H'_t \mathbf{x}_t + \mathbf{e}_t \end{cases}, t \in \mathbb{Z}, \quad (1)$$

where $\{\mathbf{x}_t\}$, $\{\mathbf{y}_t\}$, $\{\mathbf{e}_t\}$ and $\{\epsilon_t\}$ are random processes of dimensions $r \times 1$, $m \times 1$, $m \times 1$, and $d \times 1$ respectively, with

$$\begin{cases} E(\epsilon_t) = E(\mathbf{e}_t) = 0 \\ E(\epsilon_t \epsilon'_{t+h}) = \delta_{h,0} Q_t \\ E(\mathbf{e}_t \mathbf{e}'_{t+h}) = \delta_{h,0} R_t \end{cases} \quad \text{and} \quad \begin{cases} E(\epsilon_t \mathbf{x}'_{t-k}) = 0 \\ E(\mathbf{e}_t \mathbf{y}'_{t-k}) = 0 \\ E(\mathbf{x}_t \mathbf{x}'_t) = W_t \end{cases}, \quad \begin{matrix} \forall t, h \in \mathbb{Z} \\ \forall k \geq 0 \end{matrix},$$

(δ stands for the Kronecker function). The nonrandom matrices F_t , G_t , H'_t , Q_t , R_t , and W_t are periodic in time with period S . To simplify the exposition we suppose without loss of generality that

$$E(\mathbf{e}_t \epsilon'_l) = 0, \quad \forall t, l \in \mathbb{Z}.$$

Let $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{y}}_t$ be the linear least squares forecasts of \mathbf{x}_t and \mathbf{y}_t , respectively, based on $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{t-1}$.

Then as is well known, $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{y}}_t$ may be uniquely obtained from the Kalman filter (Kalman, 1960)

which is given by the following recursions

$$\begin{cases} (a) & \Omega_t = H'_t \Sigma_t H_t + R_t, \\ (b) & K_t = F_t \Sigma_t H'_t, \\ (c) & \hat{\mathbf{y}}_t = H'_t \hat{\mathbf{x}}_t, \\ (d) & \hat{\mathbf{x}}_{t+1} = (F_t - K_t H'_t) \hat{\mathbf{x}}_t + K_t \mathbf{y}_t, \\ (e) & \Sigma_{t+1} = F_t \Sigma_t F'_t - K_t \Omega_t^{-1} K'_t + G_t Q_t G'_t, \end{cases} \quad (2)$$

with starting values

$$\begin{cases} (f) & \hat{\mathbf{x}}_1 = E(\mathbf{x}_1) = 0, \\ (g) & \Sigma_1 = E(\mathbf{x}_1 \mathbf{x}_1') = W_1, \end{cases}$$

where $\hat{\mathbf{e}}_t = \mathbf{y}_t - \hat{\mathbf{y}}_t$ is the \mathbf{y}_t -residuals with covariance matrix Ω_t , $\Sigma_t = E[(\mathbf{x}_t - \hat{\mathbf{x}}_t)(\mathbf{x}_t - \hat{\mathbf{x}}_t)']$ is interpreted as the covariance matrix of the one-step state prediction errors, and $K_t = E(\mathbf{x}_{t+1} \hat{\mathbf{e}}_t')$ is known as the Kalman gain. The notation $A \geq 0$ means that the matrix A is nonnegative definite.

Recursion (2e) based on the starting equation (2g) will be called *periodic Riccati difference equation (PRDE)* because in the limit, i.e. when Σ_{t+Sk} converges as $k \rightarrow \infty$ for all $t \in \{1, \dots, S\}$, the S -periodic limiting solution $P_t = \lim_{k \rightarrow \infty} \Sigma_{t+Sk}$ will satisfy the following *discrete-time matrix periodic Riccati equation (DPRE)*

$$P_{t+1} = F_t P_t F_t' - F_t P_t H_t (H_t' P_t H_t + R_t)^{-1} H_t' P_t F_t' + G_t Q_t G_t', \quad t \in \{1, \dots, S\},$$

which has been extensively studied (see for example Bittanti et al, 1988 for some theoretical aspects and Hench and Laub, 1994 for a numerical resolution). As is well known, the resolution of (2e) requires $O(r^3)$ operations per iteration which is computationally expensive. Furthermore, the solution Σ_t must be nonnegative definite, a property that is not easy to preserve in a numerical resolution of (2e). The following section proposes some recursions that avoid these drawbacks and may have further advantages over the Kalman filter (2).

II. PERIODIC CHANDRASEKHAR-TYPE ALGORITHMS

The recursions proposed in this section and which are aimed to generalize Morf et al's (1974) algorithms to the periodic case will be called analogously periodic Chandrasekhar-type equations. This, of course, will not mean that there is an analog of our recursions in the periodic continuous-time case. The derivation of our recursions is similar to its classical counterpart and is based on the factorization result given below (see *Theorem 3.1*).

Let $\Delta_S \Sigma_t = \Sigma_{t+S} - \Sigma_t$ denote the S -lagged increment of the Riccati variable, for given $\Sigma_1, \Sigma_2, \dots, \Sigma_S \geq 0$. Then, one can prove the following result.

Theorem 3.1 *The S -lagged increment $\Delta_S \Sigma_t$ satisfies the following difference equations*

$$\Delta_S \Sigma_{t+1} = (F_t - K_{t+S} \Omega_{t+S}^{-1} H_t') [\Delta_S \Sigma_t + \Delta_S \Sigma_t H_t \Omega_t^{-1} H_t' \Delta_S \Sigma_t] (F_t - K_{t+S} \Omega_{t+S}^{-1} H_t')', \quad (3)$$

$$\Delta_S \Sigma_{t+1} = (F_t - K_t \Omega_t^{-1} H_t') [\Delta_S \Sigma_t - \Delta_S \Sigma_t H_t \Omega_{t+S}^{-1} H_t' \Delta_S \Sigma_t] (F_t - K_t \Omega_t^{-1} H_t')'. \quad (4)$$

Proof

i) Proof of (3)

From (2a) we have

$$\begin{aligned} \Omega_{t+S} &= H_t' \Delta_S \Sigma_t H_t + H_t' \Sigma_t H_t + R_t \\ &= H_t' \Delta_S \Sigma_t H_t + \Omega_t. \end{aligned}$$

Hence

$$\Omega_{t+S} = \Omega_t + \Delta_S \Omega_t, \quad (5)$$

where $\Delta_S \Omega_t \stackrel{\text{def}}{=} H_t' \Delta_S \Sigma_t H_t$. Moreover, from (2e) it follows that

$$\begin{aligned} \Sigma_{t+1+S} &= F_t \Sigma_{t+S} F_t' - \tilde{K}_{t+S} \Omega_{t+S} \tilde{K}_{t+S}' + G_t Q_t G_t', \\ \Sigma_{t+1} &= F_t \Sigma_t F_t' - \tilde{K}_t \Omega_t \tilde{K}_t' + G_t Q_t G_t', \end{aligned}$$

where $\tilde{K}_t = K_t \Omega_t^{-1}$. Therefore,

$$\Delta_S \Sigma_{t+1} = F_t \Delta_S \Sigma_t F_t' - \tilde{K}_{t+S} \Omega_{t+S} \tilde{K}_{t+S}' + \tilde{K}_t \Omega_t \tilde{K}_t'. \quad (6)$$

On the other hand, the Kalman gain \tilde{K}_t may be written in a backward recursive form as follows

$$\begin{aligned}
\tilde{K}_t &= (F_t \Sigma_{t+S} H_t - F_t \Delta_S \Sigma_t H_t) \Omega_t^{-1} \\
&= \left(\tilde{K}_{t+S} \Omega_{t+S} - F_t \Delta_S \Sigma_t H_t \right) \Omega_t^{-1} \\
&= \left[\tilde{K}_{t+S} \left(H_t' \Sigma_{t+S} H_t + R_t \right) - F_t \Delta_S \Sigma_t H_t \right] \Omega_t^{-1} \\
&= \left[\tilde{K}_{t+S} \left(H_t' \Delta_S \Sigma_t H_t + H_t' \Sigma_t H_t + R_t \right) - F_t \Delta_S \Sigma_t H_t \right] \Omega_t^{-1} \\
&= \left[\tilde{K}_{t+S} \Omega_t + \tilde{K}_{t+S} H_t' \Delta_S \Sigma_t H_t - F_t \Delta_S \Sigma_t H_t \right] \Omega_t^{-1} \\
&= \tilde{K}_{t+S} - \left(F_t - \tilde{K}_{t+S} H_t' \right) \Delta_S \Sigma_t H_t \Omega_t^{-1}.
\end{aligned}$$

Whence

$$\tilde{K}_t = \tilde{K}_{t+S} - \Delta_S \tilde{K}_t, \quad (7)$$

with $\Delta_S \tilde{K}_t \stackrel{def}{=} \left(F_t - \tilde{K}_{t+S} H_t' \right) \Delta_S \Sigma_t H_t \Omega_t^{-1}$.

Now replacing the latter expression of \tilde{K}_t in the last term of the right hand side of (6) while using (5), we obtain

$$\begin{aligned}
\Delta_S \Sigma_{t+1} &= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_{t+S} \Omega_{t+S} \tilde{K}_{t+S}' + \tilde{K}_t \Omega_t \tilde{K}_t' \\
&= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_{t+S} \left(\Omega_t + H_t' \Delta_S \Sigma_t H_t \right) \tilde{K}_{t+S}' + \left(\tilde{K}_{t+S} - \Delta_S \tilde{K}_t \right) \Omega_t \left(\tilde{K}_{t+S} - \Delta_S \tilde{K}_t \right)' \\
&= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_{t+S} \Omega_t \tilde{K}_{t+S}' - \tilde{K}_{t+S} H_t' \Delta_S \Sigma_t H_t \tilde{K}_{t+S}' \\
&\quad + \tilde{K}_{t+S} \Omega_t \tilde{K}_{t+S}' - \tilde{K}_{t+S} \Omega_t \Delta_S \tilde{K}_t' - \Delta_S \tilde{K}_t \Omega_t \tilde{K}_{t+S}' + \Delta_S \tilde{K}_t \Omega_t \Delta_S \tilde{K}_t' \\
&= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_{t+S} H_t' \Delta_S \Sigma_t H_t \tilde{K}_{t+S}' - \tilde{K}_{t+S} H_t' \Delta_S \Sigma_t \left(F_t - \tilde{K}_{t+S} H_t' \right)' \\
&\quad - \left(F_t - \tilde{K}_{t+S} H_t' \right) \Delta_S \Sigma_t H_t \tilde{K}_{t+S}' + \left(F_t - \tilde{K}_{t+S} H_t' \right) \Delta_S \Sigma_t H_t \Omega_t^{-1} H_t' \Delta_S \Sigma_t \left(F_t - \tilde{K}_{t+S} H_t' \right)' \\
&= \left(F_t - \tilde{K}_{t+S} H_t' \right) \Delta_S \Sigma_t F_t' - \left(F_t - \tilde{K}_{t+S} H_t' \right) \Delta_S \Sigma_t H_t \tilde{K}_{t+S}' \\
&\quad + \left(F_t - \tilde{K}_{t+S} H_t' \right) \Delta_S \Sigma_t H_t \Omega_t^{-1} H_t' \Delta_S \Sigma_t \left(F_t - \tilde{K}_{t+S} H_t' \right)' \\
&= \left(F_t - \tilde{K}_{t+S} H_t' \right) \left[\Delta_S \Sigma_t + \Delta_S \Sigma_t H_t \Omega_t^{-1} H_t' \Delta_S \Sigma_t \right] \left(F_t - \tilde{K}_{t+S} H_t' \right)',
\end{aligned}$$

proving (3).

ii) **Proof of (4)**

A similar argument may be used to prove (4). It suffice to express \tilde{K}_{t+S} with respect of \tilde{K}_t in a forward recursive form as follows

$$\begin{aligned}
\tilde{K}_{t+S} &= (F_t \Sigma_t H_t + F_t \Delta_S \Sigma_t H_t) \Omega_{t+S}^{-1} \\
&= \left(\tilde{K}_t \Omega_t + F_t \Delta_S \Sigma_t H_t \right) \Omega_{t+S}^{-1} \\
&= \left[\tilde{K}_t \left(H_t' \Sigma_t H_t + R_t \right) + F_t \Delta_S \Sigma_t H_t \right] \Omega_{t+S}^{-1} \\
&= \left[\tilde{K}_t \left(H_t' \Sigma_{t+S} H_t - H_t' \Delta_S \Sigma_t H_t + R_t \right) + F_t \Delta_S \Sigma_t H_t \right] \Omega_{t+S}^{-1} \\
&= \left[\tilde{K}_t \Omega_{t+S} - \tilde{K}_t H_t' \Delta_S \Sigma_t H_t + F_t \Delta_S \Sigma_t H_t \right] \Omega_{t+S}^{-1} \\
&= \tilde{K}_t + \left(F_t - \tilde{K}_t H_t' \right) \Delta_S \Sigma_t H_t \Omega_{t+S}^{-1},
\end{aligned}$$

that is

$$\tilde{K}_{t+S} = \tilde{K}_t + \Delta_S \tilde{K}_t, \quad (8)$$

where $\Delta_S \tilde{K}_t = \left(F_t - \tilde{K}_t H_t' \right) \Delta_S \Sigma_t H_t \Omega_{t+S}^{-1}$.

Then, replacing the expression of \tilde{K}_{t+S} given by (8) in the second term of the right hand side of (6), it follows that

$$\begin{aligned}
\Delta_S \Sigma_{t+1} &= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_{t+S} \Omega_{t+S} \tilde{K}_{t+S}' + \tilde{K}_t \Omega_t \tilde{K}_t' \\
&= F_t \Delta_S \Sigma_t F_t' - \left(\tilde{K}_t + \Delta_S \tilde{K}_t \right) \Omega_{t+S} \left(\tilde{K}_t + \Delta_S \tilde{K}_t \right)' \\
&\quad + \tilde{K}_t \left(\Omega_{t+S} - H_t' \Delta_S \Sigma_t H_t \right) \tilde{K}_t' \\
&= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_t \Omega_{t+S} \tilde{K}_t' - \tilde{K}_t \Omega_{t+S} \Delta_S \tilde{K}_t' - \Delta_S \tilde{K}_t \Omega_{t+S} \tilde{K}_t' - \Delta_S \tilde{K}_t \Omega_{t+S} \Delta_S \tilde{K}_t' \\
&\quad + \tilde{K}_t \Omega_{t+S} \tilde{K}_t' - \tilde{K}_t H_t' \Delta_S \Sigma_t H_t \tilde{K}_t' \\
&= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_t \Omega_{t+S} \Delta_S \tilde{K}_t' - \Delta_S \tilde{K}_t \Omega_{t+S} \tilde{K}_t' - \Delta_S \tilde{K}_t \Omega_{t+S} \Delta_S \tilde{K}_t' - \tilde{K}_t H_t' \Delta_S \Sigma_t H_t \tilde{K}_t'.
\end{aligned}$$

Finally, using again (8) we can write $\Delta_S \Sigma_{t+1}$ as follows

$$\begin{aligned}
\Delta_S \Sigma_{t+1} &= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_t \left((F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \right)' - (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \tilde{K}_t' \\
&\quad - \left((F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \right) \Omega_{t+S}^{-1} \left((F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \right)' - \tilde{K}_t H_t' \Delta_S \Sigma_t H_t \tilde{K}_t' \\
&= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_t H_t' \Delta_S \Sigma_t (F_t - \tilde{K}_t H_t')' - (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \tilde{K}_t' \\
&\quad - (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \Omega_{t+S}^{-1} H_t' \Delta_S \Sigma_t (F_t - \tilde{K}_t H_t')' - \tilde{K}_t H_t' \Delta_S \Sigma_t H_t \tilde{K}_t' \\
&= F_t \Delta_S \Sigma_t F_t' - \tilde{K}_t H_t' \Delta_S \Sigma_t F_t' + \tilde{K}_t H_t' \Delta_S \Sigma_t (\tilde{K}_t H_t')' - F_t \Delta_S \Sigma_t H_t \tilde{K}_t' \\
&\quad - (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \Omega_{t+S}^{-1} H_t' \Delta_S \Sigma_t (F_t - \tilde{K}_t H_t')' \\
&= (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t F_t' - (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t (\tilde{K}_t H_t')' \\
&\quad - (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \Omega_{t+S}^{-1} H_t' \Delta_S \Sigma_t (F_t - \tilde{K}_t H_t')' \\
&= (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t (F_t - \tilde{K}_t H_t')' - (F_t - \tilde{K}_t H_t') \Delta_S \Sigma_t H_t \Omega_{t+S}^{-1} H_t' \Delta_S \Sigma_t (F_t - \tilde{K}_t H_t')',
\end{aligned}$$

showing (4). ■

Theorem 3.1 shows that $\Delta_S \Sigma_t$ may be factorized as follows

$$\Delta_S \Sigma_t = Y_t M_t Y_t', \quad (9)$$

where M_t is a square symmetric matrix, non necessarily nonnegative definite, of dimension $\text{rank}(\Delta_S \Sigma_1)$, which is at least equal to $\text{rank}(\Delta_S \Sigma_t)$. Indeed, from (3) we have

$$\text{rank}(\Delta_S \Sigma_{t+1}) \leq \text{rank}(\Delta_S \Sigma_t) \leq \dots \leq \text{rank}(\Delta_S \Sigma_1) \leq r.$$

This can be exploited to derive some recursions with a best computational complexity than the filter (2).

Let us remark that *Theorem 3.1* is not surprising since one can always write a periodically time varying state-space model (1) as a time-invariant state space model (see Meyer and Burrus, 1975) to which it may be possible to apply the standard Chandrasekhar type factorization due to Morf et al (1974). Nevertheless, because of the requiring increasing bookkeeping (the obtained time

invariant system is of dimension multiplied by S) the development of a proper theory for periodic state-space models would be fruitful.

Thanks to the factorization result given by *Theorem 3.1*, the matrices Y_t and M_t given by (9) can be obtained recursively. The following algorithm shows that the periodic Riccati difference equation (2e) may be replaced by a set of recursions on Ω_t , K_t , Y_t and M_t with a reduction in computational efforts, especially when the state dimension r is much larger than m , the dimension of y_t .

Algorithm 3.1 *The Kalman filter (2) can be replaced by a set of recursive equations containing (2c) and (2d) and the following recursions*

$$\left\{ \begin{array}{l} (a) \quad \Omega_{t+S} = \Omega_t + H_t' Y_t M_t Y_t' H_t, \\ (b) \quad K_{t+S} = (K_t + F_t Y_t M_t Y_t' H_t), \\ (c) \quad Y_{t+1} = (F_t - K_{t+S} \Omega_{t+S}^{-1} H_t') Y_t, \\ (d) \quad M_{t+1} = M_t + M_t Y_t' H_t \Omega_t^{-1} H_t' Y_t M_t, \end{array} \right. \quad (10)$$

with starting values

$$\left\{ \begin{array}{l} (e) \quad \Omega_s = H_s' \Sigma_s H_s, \quad s = 1, \dots, S, \\ (f) \quad K_s = F_s \Sigma_s H_s, \quad s = 1, \dots, S, \end{array} \right.$$

where Σ_s , $1 \leq s \leq S$ is determined from (2e) and (2g), while Y_1 and M_1 are obtained by factorizing nonuniquely

$$\Delta_S \Sigma_1 = F_S \Sigma_S F_S' - K_S \Omega_S^{-1} K_S' + G_S Q_S G_S' - \Sigma_1, \quad (10g)$$

as

$$Y_1 M_1 Y_1'.$$

Derivation (10a) is just (6) when using (9), while (10b) follows from (9) and the relation

$$K_{t+S} = (F_t \Sigma_t H_t + F_t \Delta_S \Sigma_t H_t).$$

On the other hand, from (3) which we rewrite while using (9) we obtain

$$\begin{aligned}
\Delta_S \Sigma_{t+1} &= \left(F_t - \tilde{K}_{t+S} H_t' \right) \left[\Delta_S \Sigma_t + \Delta_S \Sigma_t H_t \Omega_t^{-1} H_t' \Delta_S \Sigma_t \right] \left(F_t - \tilde{K}_{t+S} H_t' \right)' \\
&= \left(F_t - \tilde{K}_{t+S} H_t' \right) \left[Y_t M_t Y_t' + Y_t M_t Y_t' H_t \Omega_t^{-1} H_t' Y_t M_t Y_t' \right] \left(F_t - \tilde{K}_{t+S} H_t' \right)' \\
&= \left(F_t - K_{t+S} \Omega_{t+S}^{-1} H_t' \right) Y_t \left(M_t + M_t Y_t' H_t \Omega_t^{-1} H_t' Y_t M_t \right) Y_t' \left(F_t - K_{t+S} \Omega_{t+S}^{-1} H_t' \right)' \\
&= Y_{t+1} M_{t+1} Y_{t+1}'.
\end{aligned}$$

By simple identification we get (10c) and (10d). ■

Note that the *PRDE* (2e) must be executed for $1 \leq s \leq S$ to start recursions (10). However, for $t > S$ the recursive calculation of Σ_t is not dealt with by the above algorithm but can be deduced from it through the following equation

$$\Sigma_{kS+s} = \Sigma_s + \sum_{j=0}^{k-1} Y_{jS+s} M_{jS+s} Y_{jS+s}', \quad 1 \leq s \leq S.$$

Similarly to the time-invariant case (Morf et al, 1974), other forms of *Algorithm 3.1* can be derived from *Theorem 3.1*. The following variant is particularly well adapted when $M_1 < 0$, in which case we have $M_t \leq 0$ for any t . This case is encountered whenever the periodic state-space model (1) is periodically stationary (causal) as we can see below.

Algorithm 3.2 The following set of recursions in which (10a), (10b) and (10e)-(10g) (3.8a) are unchanged while (10c) and (10d) are replaced by

$$\begin{cases} (a) & Y_{t+1} = (F_t - K_t \Omega_t^{-1} H_t') Y_t, \\ (b) & M_{t+1} = M_t - M_t Y_t' H_t \Omega_{t+S}^{-1} H_t' Y_t M_t, \end{cases} \quad (11)$$

provides the same results as *Algorithm 3.1*.

Derivation The derivation is similar to that of *Algorithm 3.1*, but is based on the factorization (4) rather than (3). ■

It is still possible to derive other forms similarly to the standard time-invariant case. The homogenous periodic Riccati difference equation (10d) can be linearized using the matrix inversion

lemma (Morf et al, 1974) through which, we obtain a recursion on M_t^{-1} rather than on M_t as follows

$$M_{t+1}^{-1} = M_t^{-1} - Y_t' H_t \Omega_{t+S}^{-1} H_t' Y_t.$$

It is worth noting that the periodic Chandrasekhar recursions given by *Algorithm 3.1* and *Algorithm 3.2* will be preferred to the Kalman filter (2) whenever the dimension of Y_t and/or M_t are significantly less than that of Σ_t . These dimensions are conditioned on the good choice of the factorization $\Delta_S \Sigma_1 = Y_1 M_1 Y_1'$ in the initialization step which will be studied in the following section.

III. THE INITIALIZATION PROBLEM

As is well known, the most important step in the development of a Chandrasekhar algorithm is the initialization step because it modulates the computational complexity and hence the lack of numerical advantage over the Kalman filter. In our periodic case, this step depends on the relation between the period S , the output dimension m , and the state dimension r . First of all, suppose the process $\{\mathbf{x}_t\}$ given by (1) is periodically stationary, that is, all the eigenvalues of the monodromy matrix $\prod_{i=0}^S F_{S-i}$ are less than unity in modulus. Let us consider two cases.

i) Case where $Sm < r$:

As pointed out in (10g) the start up values Y_1 and M_1 are determined by factorizing $\Delta_S \Sigma_1$ as $Y_1 M_1 Y_1'$. Iterating (10g) S times as follows

$$\begin{aligned} \Delta_S \Sigma_1 &= F_S \Sigma_S F_S' - \tilde{K}_S \Omega_S \tilde{K}_S' + G_S Q_S G_S' - \Sigma_1 \\ &= F_S \left(F_{S-1} \Sigma_{S-1} F_{S-1}' - \tilde{K}_{S-1} \Omega_{S-1} \tilde{K}_{S-1}' + G_{S-1} Q_{S-1} G_{S-1}' \right) F_S' \\ &\quad - \tilde{K}_S \Omega_S \tilde{K}_S' + G_S Q_S G_S' - \Sigma_1 \\ &= F_S F_{S-1} \Sigma_{S-1} (F_S F_{S-1})' - F_S \tilde{K}_{S-1} \Omega_{S-1} \tilde{K}_{S-1}' F_S' + F_S G_{S-1} Q_{S-1} G_{S-1}' F_S' \\ &\quad - \tilde{K}_S \Omega_S \tilde{K}_S' + G_S Q_S G_S' - \Sigma_1 \end{aligned}$$

$$\begin{aligned}
&= F_S F_{S-1} \left(F_{S-2} \Sigma_{S-2} F'_{S-2} - \tilde{K}_{S-2} \Omega_{S-2} \tilde{K}'_{S-2} + G_{S-2} Q_{S-2} G'_{S-2} \right) (F_S F_{S-1})' \\
&\quad - F_S \tilde{K}_{S-1} \Omega_{S-1} \tilde{K}'_{S-1} F'_S + F_S G_{S-1} Q_{S-1} G'_{S-1} F'_S - \tilde{K}_S \Omega_S \tilde{K}'_S + G_S Q_S G'_S - \Sigma_1 \\
&= F_S F_{S-1} F_{S-2} \Sigma_{S-2} (F_S F_{S-1} F_{S-2})' - F_S F_{S-1} \tilde{K}_{S-2} \Omega_{S-2} (F_S F_{S-1} \tilde{K}_{S-2})' \\
&\quad + (F_S F_{S-1}) G_{S-2} Q_{S-2} G'_{S-2} (F_S F_{S-1})' \\
&\quad - F_S \tilde{K}_{S-1} \Omega_{S-1} (\tilde{K}_{S-1} F_S)' + F_S G_{S-1} Q_{S-1} G'_{S-1} F'_S - \tilde{K}_S \Omega_S \tilde{K}'_S + G_S Q_S G'_S - \Sigma_1 \\
&\quad \vdots \\
&= - \sum_{k=0}^{S-1} \left(\prod_{j=0}^{k-1} F_{S-j} \right) \tilde{K}_{S-k} \Omega_{S-k} \tilde{K}'_{S-k} \left(\prod_{j=0}^{k-1} F_{S-j} \right)' + \left(\prod_{j=0}^{S-1} F_{S-j} \right) \Sigma_1 \left(\prod_{j=0}^{S-1} F_{S-j} \right)' \\
&\quad + \sum_{k=0}^{S-1} \left(\prod_{j=0}^{k-1} F_{S-j} \right) G_{S-k} Q_{S-k} G'_{S-k} \left(\prod_{j=0}^{k-1} F_{S-j} \right)' - \Sigma_1, \tag{12}
\end{aligned}$$

and invoking the fact that under the periodic stationarity assumption, Σ_1 satisfies the following discrete-time periodic Lyapunov equation (*DPLE*) (e.g. Bittanti et al, 1988; Varga, 1997)

$$\Sigma_1 = \left(\prod_{j=0}^{S-1} F_{S-j} \right) \Sigma_1 \left(\prod_{j=0}^{S-1} F_{S-j} \right)' + \sum_{k=0}^{S-1} \left(\prod_{j=0}^{k-1} F_{S-j} \right) G_{S-k} Q_{S-k} G'_{S-k} \left(\prod_{j=0}^{k-1} F_{S-j} \right)',$$

we conclude that the sum of the last three terms of the right hand-side of (12) is zero.

Whence

$$\begin{aligned}
\Delta_S \Sigma_1 &= - \sum_{k=0}^{S-1} \left(\prod_{j=0}^{k-1} F_{S-j} \right) \tilde{K}_{S-k} \Omega_{S-k} \tilde{K}'_{S-k} \left(\prod_{j=0}^{k-1} F_{S-j} \right)' \\
&= - \sum_{k=0}^{S-1} \left(\prod_{j=0}^{k-1} F_{S-j} \right) K_{S-k} \Omega_{S-k}^{-1} K'_{S-k} \left(\prod_{j=0}^{k-1} F_{S-j} \right)' \\
&= -L \begin{pmatrix} \Omega_S^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Omega_1^{-1} \end{pmatrix} L' = Y_1 M_1 Y_1', \tag{13}
\end{aligned}$$

where L is given by

$$L = \left[K_S, F_S K_{S-1}, F_S F_{S-1} K_{S-2}, \dots, \prod_{j=0}^{S-1} F_{S-j} K_1 \right].$$

Clearly, with such a factorization the dimension of M_1 (and hence of M_t for every t) is equal to mS which is less than r , the dimension of the Riccati matrix associated with the Kalman filter (2). Indeed, when Sm is fairly less than r , the nonhomogeneous $PRDE$ (2e) may be replaced by the homogenous $PRDE$ (11b) which is of lower dimension. For instance, for $m = 1$, the complexity of solving (10d) or (11b) when using (13) as an initialization step is of order $O(Sr^2)$ which is computationally simple to solve compared to the $PRDE$ (2e). It is still possible to improve the computation of (13) by alleviating the formation of the sums of products in L by using the periodic Schur decomposition (Bojanczyk et al, 1992; Hensch and Laub, 1994).

ii) **Case where** $Sm \geq r$:

In this case the latter factorization given by (13) would be inefficient since the dimension of M_t is greater than that of Σ_t . Thus we have to search for another factorization. We have

$$\begin{aligned}
 \Sigma_1 &= E(x_1 - \hat{x}_1)(x_1 - \hat{x}_1)' = E(x_1 x_1') \\
 &= E(F_S x_0 + G_S w_0)(F_S x_0 + G_S w_0)' \\
 &= F_S E(x_0 x_0') F_S' + G_S E(w_0 w_0') G_S' \\
 &= F_S W_0 F_S' + G_S Q_S G_S'.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \Delta_S \Sigma_1 &= Y_1 M_1 Y_1' \\
 &= F_S \Sigma_S F_S' - \tilde{K}_S \Omega_S \tilde{K}_S' + G_S Q_S G_S' - F_S W_0 F_S' - G_S Q_S G_S' \\
 &= F_S (\Sigma_S - W_0) F_S' - \tilde{K}_S \Omega_S \tilde{K}_S' \\
 &= F_S (\Sigma_S - W_0) F_S' - (F_S \Sigma_S H_S \Omega_S^{-1}) \Omega_S (F_S \Sigma_S H_S \Omega_S^{-1})' \\
 &= F_S \left[\Sigma_S - W_0 - (\Sigma_S H_S) \Omega_S^{-1} (\Sigma_S H_S)' \right] F_S'.
 \end{aligned}$$

This allows to identify Y_1 and M_1 as follows

$$\begin{cases} Y_1 = F_S \\ M_1 = \Sigma_S - W_0 - (\Sigma_S H_S) \Omega_S^{-1} (\Sigma_S H_S)' . \end{cases} \quad (14)$$

With such an initialization, the *PRDE* (11b) has the same dimension as that of the *PRDE* (2e), and it seems that there is no reduction in the computational cost compared to the Kalman filter. However, the difference from (2e) is that, unlike the Σ_t , the M_t is not required to be nonnegative-definite. This helps alleviate the computational complexity of (10d) and then (11b).

In the matter of illustration we propose the following example which shows the impact of a good choice of a starting factorization on the Chandrasekhar algorithm complexity.

Example 4.1 Consider a periodic autoregression of order 5 and period S ($PAR_S(5)$), which is given by the following stochastic difference equation

$$y_t - \phi_1^{(t)} y_{t-1} - \phi_2^{(t)} y_{t-2} - \phi_3^{(t)} y_{t-3} - \phi_4^{(t)} y_{t-4} - \phi_5^{(t)} y_{t-5} = \varepsilon_t, \quad (15)$$

where $\{\varepsilon_t\}$ is a periodic white noise with S -periodic variance and where the parameters $\phi_j^{(t)}$, $j = 1, \dots, 5$ are periodic with respect of t with S .

Setting $\mathbf{x}_t = (y_t, y_{t-1}, \dots, y_{t-4})'$, $\varepsilon_t = (\varepsilon_t, 0, 0, 0, 0)'$ and $H' = (1, 0, 0, 0, 0)'$, model (15) may be written in the state-space form

$$\begin{aligned} \mathbf{x}_t &= F_t \mathbf{x}_{t-1} + \varepsilon_t \\ y_t &= H' \mathbf{x}_t \end{aligned} \quad (16)$$

so that identifying it with model (1), the dimensions r , m and d are respectively equal to 5, 1 and 5.

When applying the Kalman filter to model (16), the corresponding periodic Riccati equation (2e) is of dimension 5 (the dimension of Σ_t) for any value of S . However, the dimension of the Riccati equation corresponding to the periodic Chandrasekhar filter (dimension of M_t) depends upon S .

Let us consider two cases for S .

i) **Case where $S = 2$.**

We are in the case where $Sm < r$. According to formula (13), we have

$$\Delta_S \Sigma_1 = -L \begin{pmatrix} \Omega_2^{-1} & 0 \\ 0 & \Omega_1^{-1} \end{pmatrix} L' = Y_1 M_1 Y_1',$$

with $L = [K_2, F_2 K_1]$, $K_1 = F_1 \Sigma_1 H_1$ and $\Omega_t = H_t' \Sigma_t H_t$, $t = 1, 2$. So, we can take $M_1 = \begin{pmatrix} \frac{1}{\Omega_2} & 0 \\ 0 & \frac{1}{\Omega_1} \end{pmatrix}$, from which the corresponding Riccati equation is of dimension 2, clearly lower than the dimension of the Riccati equation of the Kalman filter. Whence in this case the periodic Chandrasekhar filter is highly superior to its homologue, the Kalman one.

ii) **Case where $S = 12$.**

In this case, the previous factorization is inefficient since the dimension of the Chandrasekhar Riccati would be equal to 12, much larger than 5, the dimension of the Kalman Riccati. Nevertheless, we are in the case $Sm > r$, and according to (14), $\Delta_S \Sigma_1$ may be factorized as $Y_1 M_1 Y_1'$, where

$$Y_1 = F_{12} \text{ and } M_1 = \Sigma_{12} - W_0 - \Sigma_{12} H_{12} \Omega_{12}^{-1} H_{12}' \Sigma_{12}',$$

so that the Riccati equation associated with the Chandrasekhar filter has the same dimension as that of the Riccati equation of the Kalman filter. Moreover, the matrix M_1 is not necessarily nonnegative definite in contrast with Σ_1 , and from this viewpoint the Chandrasekhar filter is still more suitable.

IV. CONCLUSION

In this paper the discrete-time Chandrasekhar recursions have been generalized to the periodic time-varying state-space case through several forms. These recursions allow in a large range of cases to solve the periodic Riccati difference equation with a considerable reduction in the computational complexity. Along similar lines to the standard time-invariant case (Morf and Kailath, 1975), a square root version of these recursions may be easily derived in order to improve the numerical stability of the proposed algorithms. Useful applications for time series analysis as well as for the periodic system theory can be given, in particular, we mention the likelihood evaluation of periodic

VARMA (Aknouche and Hamdi, 2007), the calculation of exact Fisher information matrix for *PARMA* models and the development of fast *RLS* algorithms for periodic systems (Bentarzi and Aknouche, 2006).

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